The Stretch Factor of L_1- and L_∞-Delaunay Triangulations

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Abstract

In this paper we determine the stretch factor of the L_1-Delaunay and L_∞-Delaunay triangulations, and we show that this stretch is $\sqrt{4+2\sqrt{2}} \approx 2.61$. Between any two points x, y of such triangulations, we construct a path whose length is no more than $\sqrt{4+2\sqrt{2}}$ times the Euclidean distance between x and y, and this bound is best possible. This definitively improves the 25-year old bound of $\sqrt{10}$ by Chew (SoCG '86).

To the best of our knowledge, this is the first time the stretch factor of the well-studied L_p-Delaunay triangulations, for any real $p \geq 1$, is determined exactly.

Keywords: Delaunay triangulations, L_1-metric, L_∞-metric, stretch factor

1 Introduction

Given a set of points P on the plane, the Delaunay triangulation for P is a spanning subgraph of the Euclidean graph on P that is the dual of the Voronoï diagram of P. The Delaunay triangulation is a fundamental structure with many applications in computational geometry and other areas of Computer Science. In some applications (including on-line routing [BM04]), the triangulation is used as a spanner, defined as a spanning subgraph in which the distance between any pair of points is no more than a constant multiplicative ratio of the Euclidean distance between the points. The constant ratio is typically referred to as the stretch factor of the spanner. While Delaunay triangulations have been studied extensively, obtaining a tight bound on its stretch factor has been elusive even after decades of attempts.

In the mid-1980s, it was not known whether Delaunay triangulations were spanners at all. In order to gain an understanding of the spanning properties of Delaunay triangulations, Chew considered related, "easier" structures. In his seminal 1986 paper [Che86], he proved that an L_1-Delaunay triangulation — the dual of the Voronoï diagram of P based on the L_1-metric rather than the L_2-metric — has a stretch factor bounded by $\sqrt{10}$. Chew then continued on

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Paper	**Graph**	**Stretch factor**
[DFS87]	L_2-Delaunay	$\pi(1+\sqrt{5})/2 \approx 5.08$
[KG92]	L_2-Delaunay	$4\pi/(3\sqrt{3}) \approx 2.41$
[Xia11]	L_2-Delaunay	1.998
[Che89]	TD-Delaunay	**2**
[Che86]	L_1-,L_∞-Delaunay	$\sqrt{10} \approx 3.16$
[this paper]	L_1-,L_∞-Delaunay	$\boldsymbol{\sqrt{4+2\sqrt{2}} \approx 2.61}$

Table 1: Key stretch factor upper bounds (optimal values are bold).

and showed that the a TD-Delaunay triangulation — the dual of a Voronoï diagram defined using a *Triangular Distance*, a distance function not based on a circle (L_2-metric) or a square (L_1-metric) but an equilateral triangle — has a stretch factor of 2 [Che89].

Finally, Dobkin et al. [DFS87] succeeded in showing that the (classical, L_2-metric) Delaunay triangulation of P is a spanner as well. The bound on the stretch factor they obtained was subsequently improved by Keil and Gutwin [KG92] as shown in Table 1. The bound by Keil and Gutwin stood unchallenged for many years until very recently when Xia improved the bound to below 2 [Xia11].

While progress has been made, none of the techniques developed so far lead to a tight bound on the spanning ratio. There has been some progress recently on the lower bound side. The trivial lower bound of $\pi/2 \approx 1.5707$ has recently been improved to 1.5846 [BDL+11] and then to 1.5932 [XZ11].

While much effort has been made on understanding the stretch factor of Delaunay triangulations, little has been done on the L_p-Delaunay triangulations for $p \neq 2$. Lee and Wong [LW80] show that L_1-,L_∞-Delaunay triangulations have applications in scheduling problems for 2-dimensional storage, and how to construct, for all real $p \geq 1$, Voronoï diagrams in the L_p-metric in $O(n \log n)$ time [Lee80]. Delaunay triangulations based on arbitrary convex distance functions have been studied in [BCCS08], which shows that such geometric graphs are indeed plane graphs and spanners whose stretch factor depends only on the shape of the convex body. However, due to the general approach, the bounds on the stretch factor remain loose. For instance the bounds they obtain for L_2-Delaunay triangulations are > 24.

The general picture is that, in spite of much effort, with the exception of the triangular distance the exact value of the stretch factor of Delaunay triangulations based on any convex function is unknown. In particular, the stretch factor of L_p-Delaunay triangulations is unknown for each $p \geq 1$.

Our contributions. We show that the exact stretch factor of L_1-Delaunay triangulations and L_∞-Delaunay triangulations is $\sqrt{4+2\sqrt{2}} \approx 2.61$, ultimately improving the 3.16 bound of Chew [Che86].

Technically, we use rectangular coordinates to prove the upper bound. We show that the distance in the triangulation between any source-destination pair of points lying on the border of a horizontal rectangle of length x and of depth $y \leq x$ is no more than $(1+\sqrt{2})x+y$. The stretch factor bound then simply follows. In our proof, we construct the route from the source to the destination by maintaining two possible short paths, until we reach some special point (called *inductive* point) where we can apply our main inductive hypothesis.

Despite the technical aspect of our contribution, we believe that our proof techniques may give insights into determining the stretch factor of other convex distance based Delaunay triangulations. For example, let P_k denote the convex distance function defined by a regular k-gon. We observe that the stretch factor of P_k-Delaunay triangulations is known for $k = 3, 4$ since P_3 is the triangular distance function of [Che89], and P_4 is nothing else than the L_∞-metric. Determining the stretch factor of P_k-Delaunay triangulations for larger k would undoubtedly be an important step towards understanding the stretch factor of classical Delaunay triangulations.

2 Preliminaries

Given a set P of points in the two-dimensional Euclidean space, the Euclidean graph $\mathcal{E}$ is the complete weighted graph embedded in the plane whose nodes are identified with the points. We assume a Cartesian coordinate system is associated with the Euclidean space and thus every point can be specified with x and y coordinates. For every pair of nodes u and w, the edge (u, w) represents the segment $[uw]$ and its weight is the edge length defined in Euclidean distance: $d_2(u, w) = \sqrt{d_x(u, w)^2 + d_y(u, w)^2}$ where $d_x(u, w)$ (resp. $d_y(u, w)$) is the difference between the x (resp. y) coordinates of u and w.

We say that a subgraph H of a graph G is a *t-spanner* of G if for any pair of vertices u, v of G, the distance between u and v in H is at most t times the distance between u and v in G; the constant t is referred to as the *stretch factor* of H (with respect to G). H is a t-spanner (or spanner for some t constant) if it is a t-spanner of $\mathcal{E}$.

In our paper, we deal with the construction of spanners based on Delaunay triangulations. As we saw in the introduction, the L_1-Delaunay triangulation is the dual of the Voronoï diagram based on the L_1-metric $d_1(u, w) = d_x(u, w) + d_y(u, w)$. A property of the L_1-Delaunay triangulations, actually shared by all L_p-Delaunay triangulations, is that all their triangles can be defined in terms of empty circumscribed convex bodies (squares for L_1 or L_∞ and circles for L_2). More precisely, let a *square* in the plane be a square whose sides are parallel to the x and y axis and let a *tipped square* be a square tipped at 45°. For every pair of points $u, v \in P$, (u, v) is an edge in the *L_1-Delaunay triangulation* of P iff there is a tipped square that has u and v on its boundary and contains no point of P in its interior (cf. [Che89]).

If a *square* with sides parallel to the x and y axes, rather than a tipped square, is used in this definition then a different triangulation is defined; it corresponds to the dual of the Voronoï diagram based on the L_∞-metric $d_\infty(u, w) = \max\{d_x(u, w), d_y(u, w)\}$. We refer to this triangulation as the L_∞-Delaunay triangulation. This triangulation is nothing more than the L_1-Delaunay triangulation of the set of points P after rotating all the points by 45° around the origin. Therefore Chew's bound of $\sqrt{10}$ on the stretch factor of the L_1-Delaunay triangulation ([Che86]) applies to L_∞-Delaunay triangulations as well. In the remainder of this paper, we will be referring to L_∞-Delaunay (rather than L_1) triangulations because we will be (mostly) using the L_∞-metric and squares, rather than tipped squares.

One issue with Delaunay triangulations is that there might not be a unique triangulation of a given set of points P. To insure uniqueness and keep our arguments simple, we make the usual assumption that the points in P are in *general position*, which for us means that no four points lie on the boundary of a square and no two points share the same abscissa or the same ordinate.

We end this section by giving a lower bound on the stretch factor of L_∞-Delaunay triangulations.

Proposition 1 *For every $\varepsilon > 0$, there exists a set of points P in the plane such that the L_∞-Delaunay triangulation on P has stretch factor at least*

$$\sqrt{4 + 2\sqrt{2}} - \varepsilon \ .$$

This lower bound applies, of course, to L_1-Delaunay triangulations as well. The proof of this proposition relies on the example of Figure 1.

Proof. Given $\delta > 0$, we define the set of points P as follows. Let point a be the origin and let points b, c_1, and c_2 have coordinates $(1, \sqrt{2} - 1)$, $(\delta, \sqrt{2} - 2\delta)$, and $(1 - \delta, 1 - 2\delta)$, respectively. Additional $k = \frac{\sqrt{2}-2\delta}{\delta} - 1$ points are placed on line segment $[ac_1]$ and another k on line segment $[c_2b]$ in such a way that the difference in y coordinates between successive points on a segment is δ, as shown in Figures 1. (W.l.o.g. assume that $\frac{\sqrt{2}}{\delta}$ and thus k is an integer so that this can be done.) Let $a = p_0, p_1, p_2, p_3, \ldots, p_k, p_{k+1} = c_1$ be the labels, in order as they appear when moving from a to c_1, of the points on segment $[ac_1]$ and let $c_2 = q_0, q_1, q_2, q_3, \ldots, q_{k+1} = b$ be the labels, in order as they appear when moving from c_2 to b, of the points on segment $[c_2b]$, as illustrated in Figure 1.

Consider the square S_1 of side length $1 - \delta$ and having a and p_1 on its west (left) and north sides, respectively (see Figure 1b)). Since $d_\infty(a, c_2) = d_x(a, c_2) = 1 - \delta$ and $d_\infty(p_1, c_2) = d_y(p_1, c_2) = 1 - \delta$, point c_2 is exactly the southeast vertex of square S_1. By symmetry, it

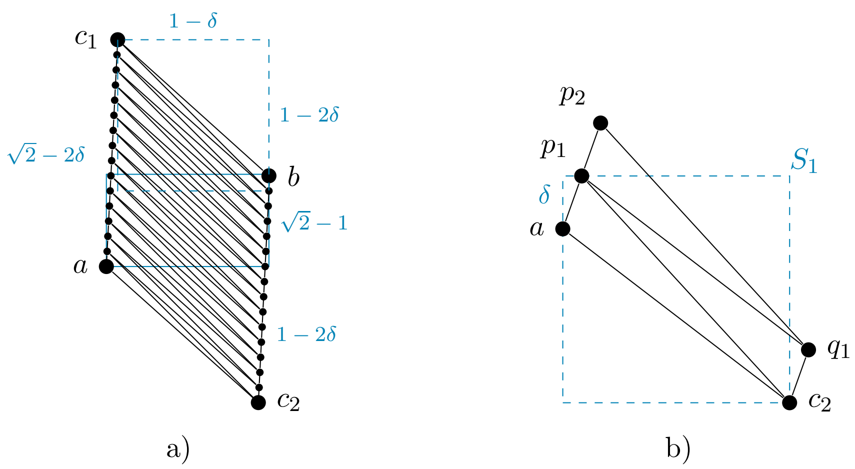

Figure 1: a) An L_∞-Delaunay triangulation with stretch factor arbitrarily close to $\sqrt{4 + 2\sqrt{2}}$. b) A closer look at the first few faces of this triangulation.

follows that for every $i = 0, 1, 2, \ldots, k$, if S_i is the square of side length $1 - \delta$ with p_i and p_{i+1} on its west and north sides, then point q_i is exactly the southeast vertex of S_i. This means that all points q_j with $j \neq i$ as well as all points p_j with $j \neq i, i+1$ must lie outside S_i. Therefore, for every $i = 0, 1, 2, \ldots, k$, points p_i, p_{i+1}, and q_i define a triangle in the L_∞-Delaunay triangulation T on P. A similar argument shows that the path $q_0, q_1, \ldots, q_{k+1}$ is in triangulation T as well. The triangulation T is illustrated in Figure 1a).

Having defined the set of points P and described its L_∞-Delaunay triangulation T, we now analyze the stretch factor of T. A shortest path from a to b in T is, for example, $a, p_1, p_2, \ldots, p_k, c_1, b$. The length of this path is

$$\begin{aligned} d_2(a, c_1) + d_2(c_1, b) &= \sqrt{d_x(a, c_1)^2 + d_y(a, c_1)^2} + \sqrt{d_x(c_1, b)^2 + d_y(c_1, b)^2} \\ &= \sqrt{(\sqrt{2} - \delta)^2 + \delta^2} + \sqrt{(1 - \delta)^2 + (1 - 2\delta)^2} \end{aligned}$$

which tends to $2\sqrt{2}$ as δ tends to 0. The Euclidean distance between a and b is:

$$d_2(a, b) = \sqrt{1^2 + (\sqrt{2} - 1)^2} = \sqrt{4 - 2\sqrt{2}}\ .$$

Therefore, it is possible to construct a L_∞-Delaunay triangulation whose stretch factor is arbirtrarily close to:

$$\frac{2\sqrt{2}}{\sqrt{4 - 2\sqrt{2}}} = \sqrt{4 + 2\sqrt{2}}\ .$$

□

3 Main result

In this section we obtain a tight upper bound on the stretch factor of an L_∞-Delaunay triangulation. It follows from this key theorem:

Theorem 2 *Let T be the L_∞-Delaunay triangulation on a set of points P in the plane and let a and b be any two points of P. If $x = d_\infty(a, b) = \max\{d_x(a, b), d_y(a, b)\}$ and $y = \min\{d_x(a, b), d_y(a, b)\}$ then*

$$d_T(a, b) \ \leq\ (1 + \sqrt{2})x + y$$

where $d_T(a, b)$ denotes the distance between a and b in triangulation T.

Corollary 3 *The stretch factor of the L_1- and the L_∞-Delaunay triangulation on a set of points P is at most*

$$\sqrt{4 + 2\sqrt{2}} \ \approx\ 2.6131259\ldots$$

Proof. By Theorem 2, an upper-bound of the stretch factor of an L_∞-Delaunay triangulation is the maximum of the function

$$\frac{(1 + \sqrt{2})x + y}{\sqrt{x^2 + y^2}}$$

over values x and y such that $0 < y \leq x$. The maximum is reached when x and y satisfy $y/x = 1 + \sqrt{2}$, and the maximum is equal to $\sqrt{1 + (1 + \sqrt{2})^2} = \sqrt{4 + 2\sqrt{2}}$. As L_1- and L_∞-Delaunay triangulations have the same stretch factor, this result also holds for L_1-Delaunay triangulations. $\square$

To prove Theorem 2, we will construct a bounded length path in T between two arbitrary points a and b of P. To simplify the notation and the discussion, we assume that point a has coordinates $(0, 0)$ and point b has coordinates (x, y) with $0 < y \leq x$. The segment $[ab]$ divides the Euclidean plane into two half-planes; a point in the same half-plane as point $(0, 1)$ is said to be *above* segment $[ab]$, otherwise it is *below*. Let $T_1, T_2, T_3, \ldots, T_k$ be the sequence of triangles of triangulation T that line segment $[ab]$ intersects when moving from a to b. Let h_1 and l_1 be the nodes of T_1 other than a, with h_1 lying above segment $[ab]$ and l_1 and lying below. Every triangle T_i, for $1 < i < k$, intersects line segment $[ab]$ twice; let h_i and l_i be the endpoints of the edge of T_i that intersects segment $[ab]$ last, when moving on segment $[ab]$ from a to b, with h_i being above and l_i being below segment $[ab]$. Note that either $h_i = h_{i-1}$ and $T_i = \triangle(h_i, l_i, l_{i-1})$ or $l_i = l_{i-1}$ and $T_i = \triangle(h_{i-1}, h_i, l_i)$, for $1 < i < k$. We also set $h_0 = l_0 = a$, $h_k = b$, and $l_k = l_{k-1}$. For $1 \leq i \leq k$, we define S_i to be the square whose sides pass through the three vertices of T_i (see Figure 2); since T is an L_∞-Delaunay triangulation, the interior of S_i is devoid of points of P. We will refer to the sides of the square using the notation: N (north), E (east), S (south), and W (west). We will also use this notation to describe the position of an edge connecting two points lying on two sides a square: for example, a WN edge connects a point on the west and a point on the N side. We will say that an edge is *gentle* if the line segment corresponding to it in the graph embedding has a slope within $[-1, 1]$; otherwise we will say that it is *steep*.

We will prove Theorem 2 by induction on the distance, using the L_∞-metric, between a and b. Let $R(a, b)$ be the rectangle with sides parallel to the x and y axes and with vertices at points a and b. If there is a point of P inside $R(a, b)$, we will easily apply induction. The case when $R(a, b)$ does not contain points of P — and in particular the points h_i and l_i for $0 < i < k$ — is more difficult and we need to develop tools to handle it. The following Lemma describes the structure of the triangles $T_1, \ldots, T_k$ when $R(a, b)$ is empty. We need some additional terminology first though: we say that a point u is *above* (resp. *below*) $R(a, b)$

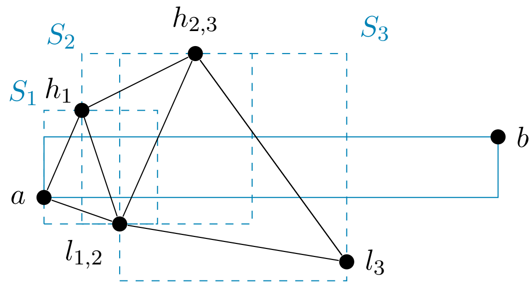

Figure 2: Triangles T_1 (with points a, h_1, l_1), T_2 (with points h_1, h_2, and l_2), and T_3 (with points l_2, h_3, and l_3) and associated squares S_1, S_2, and S_3. When traveling from a to b along segment $[a, b]$, the edge that is hit when leaving T_i is (h_i, l_i).

if $0 < x_u < x$ and $y_u > y$ (resp. $y_u < 0$).

Lemma 4 *If $(a,b) \notin T$ and no point of P lies inside rectangle $R(a,b)$, then point a lies on the W side of square S_1, point b lies on the E side of square S_k, points $h_1, \ldots, h_k$ all lie above $R(a,b)$, and points $l_1, \ldots, l_k$ all lie below $R(a,b)$. Furthermore, for any i such that $1 < i < k$:*

a) Either $T_i = \triangle(h_{i-1}, h_i, l_{i-1} = l_i)$, points h_{i-1}, h_i, and $l_{i-1} = l_i$ lie on the sides of S_i in clockwise order, and (h_{i-1}, h_i) is a WN, WE, or NE edge in S_i

b) Or $T_i = \triangle(h_{i-1} = h_i, l_{i-1}, l_i)$, points $h_{i-1} = h_i$, l_i, and l_{i-1} lie on the sides of S_i in clockwise order, and (l_{i-1}, l_i) is a WS, WE, or SE edge in S_i.

These properties are illustrated in Figure 2.

Proof. Since points of P are in general position, points a, h_1, and l_1 must lie on 3 different sides of S_1. Because segment $[ab]$ intersects the interior of S_1 and since a is the origin and b is in cone 0 of a, a can only lie on the W or S side of S_1. If a lies on the S side then $l_1 \neq b$ would have to lie inside $R(a,b)$, which is a contradiction. Therefore a lies on the W side of S_1 and, similarly, b lies on the E side of S_k.

Since points h_i $(0 < i < k)$ are above segment $[ab]$ and points l_i $(0 < i < k)$ are below segment $[ab]$, and because all squares S_i $(0 < i < k)$ intersect $[ab]$, points $h_1, \ldots, h_k$ all lie above $R(a,b)$, and points $l_1, \ldots, l_k$ all lie below $R(a,b)$.

The three vertices of T_i can be either $h_i = h_{i-1}$, l_{i-1}, and l_i or h_{i-1}, h_i, and $l_{i-1} = l_i$. Because points of T are in general position, the three vertices of T_i must appear on three different sides of S_i. Finally, because h_{i-1} and h_i are above $R(a,b)$, they cannot lie on the S side of S_i, and because l_{i-1} and l_i are below $R(a,b)$, they cannot lie on the N side of S_i.

If $T_i = \triangle(h_{i-1}, h_i, l_{i-1} = l_i)$, points h_{i-1}, h_i, l_i must lie on the sides of S_i in clockwise order. The only placements of points h_{i-1} and h_i on the sides of S_i that satisfy all these constraints are as described in *a)*. If $T_i = \triangle(h_{i-1} = h_i, l_{i-1}, l_i)$, points h_i, l_i, l_{i-1} must lie on the sides of S_i in clockwise order. Part *b)* lists the placements of points l_{i-1} and l_i that satisfy the constraints. $\square$

In the following definition, we define the points on which induction can be applied in the proof of Theorem 2.

Definition 5 *Let $R(a,b)$ be empty. Square S_j is* inductive *if edge (l_j, h_j) is gentle. The point $c = h_j$ or $c = l_j$ with the larger abcissa is the* inductive *point of inductive square S_j.*

The following lemma will be the key ingredient of our inductive proof of Theorem 2.

Lemma 6 *Assume that $R(a,b)$ is empty. If no square $S_1, \ldots, S_k$ is inductive then*

$$d_T(a,b) \leq (1+\sqrt{2})x + y \ .$$

Otherwise let S_j be the first inductive square in the sequence $S_1, S_2, \ldots, S_k$. If h_j is the inductive point of S_j then

$$d_T(a,h_j) + (y_{h_j} - y) \leq (1+\sqrt{2})x_{h_j} \ .$$

If l_j is the inductive point of S_j then

$$d_T(a, l_j) - y_{l_j} \ \leq \ (1 + \sqrt{2}) x_{l_j} \ .$$

Given an inductive point c, we can use use Lemma 6 to bound $d_T(a, b)$ and then apply induction to bound $d_T(b, c)$, *but only if* the position of point c relative to the position of point b is *good*, i.e., if $x - x_c \geq |y - y_c|$. If that is not the case, we will use the following Lemma:

Lemma 7 *Let $R(a, b)$ be empty and let the coordinates of point $c = h_i$ or $c = l_i$ satisfy $0 < x - x_c < |y - y_c|$.*

a) If $c = h_i$, and thus $0 < x - x_{h_i} < y_{h_i} - y$, then there exists j, with $i < j \leq k$ such that all edges in path $h_i, h_{i+1}, h_{i+2}, \ldots, h_j$ are NE edges in their respective squares and $x - x_{h_j} \geq y_{h_j} - y \geq 0$.

b) If $c = l_i$, and thus $0 < x - x_{l_i} < y - y_{l_i}$, then there exists j, with $i < j \leq k$ such that all edges in path $l_i, l_{i+1}, l_{i+2}, \ldots, l_j$ are SE edges and $x - x_{l_j} \geq y - y_{l_j} \geq 0$.

Proof. We only prove the case $c = h_j$ as the case $c = l_i$ follows using a symmetric argument.

We construct the path $h_i, h_{i+1}, h_{i+2}, \ldots, h_j$ iteratively. If $h_i = h_{i+1}$, we just continue building the path from h_{i+1}. Otherwise, (h_i, h_{i+1}) is an edge of T_{i+1} which, by Lemma 4, must be a WN, WE, or NE edge in square S_{i+1}. Since the S side of square S_{i+1} is below $R(a, b)$ and because $x - x_{h_i} < y_{h_i} - y$, point h_i cannot be on the W side of S_{i+1} (otherwise b would be inside square S_{i+1}). Thus (h_i, h_{i+1}) is a NE edge. If $x - x_{h_{i+1}} \geq y_{h_{i+1}} - y$ we stop, otherwise we continue the path construction from h_{i+1}. $\square$

We can now prove the main theorem.

Proof of Theorem 2. The proof is by induction on the distance, using the L_∞-metric, between points of P (since P is finite there is only a finite number of distances to consider).

Let a and b be the two points of P that are the closest points, using the L_∞-metric. We assume w.l.o.g. that a has coordinates $(0, 0)$ and b has coordinates (x, y) with $0 < y \leq x$. Since a and b are the closest points using the L_∞-metric, the largest square having a as a southwest vertex and containing no points of P in its interior, which we call S_a must have b on its E side. Therefore (a, b) is an edge in T and $d_T(a, b) = d_2(a, b) \leq x + y \leq (1 + \sqrt{2})x + y$.

For the induction step, we again assume, w.l.o.g., that a has coordinates $(0, 0)$ and b has coordinates (x, y) with $0 < y \leq x$.

Case 1: $R(a, b)$ is not empty.

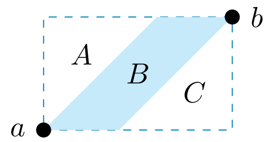

Figure 3: Partition of $R(a, b)$ into three regions in Case 1 of the proof of Theorem 2.

We first consider the case when there is at least one point of P lying inside rectangle $R(a,b)$. If there is a point c inside $R(a,b)$ such that $y_c \leq x_c$ and $y - y_c \leq x - x_c$ (i.e., c lies in the region B shown in Figure 3 then we can apply induction to get $d_T(a,c) \leq (1+\sqrt{2})x_c + y_c$ and $d_T(c,b) \leq (1+\sqrt{2})(x-x_c)+y-y_c$ and use these to obtain the desired bound for $d_T(a,b)$.

We now assume that there is no point inside region B. If there is still a point in $R(a,b)$ then there must be one that is on the border of S_a, the square we defined in the basis step, or S_b, defined as the largest square having b as a northeast vertex and containing no points of P in its interior. W.l.o.g., we assume the former and thus there is an edge $(a,c) \in T$ such that either $y_c > x_c$ (i.e., c is inside region A shown in Figure 3 or $y - y_c > x - x_c$ (i.e., c is inside region C). Either way, $d_T(a,c) = d_2(a,c) \leq x_c + y_c$. If c is in region A, since $x - x_c \geq y - y_c$, by induction we also have that $d_T(c,b) \leq (1+\sqrt{2})(x-x_c) + (y-y_c)$. Then

$$\begin{aligned} d_T(a,b) &\leq d_T(a,c) + d_T(c,b) \\ &\leq x_c + y_c + (1+\sqrt{2})(x-x_c) + (y-y_c) \leq (1+\sqrt{2})x + y \end{aligned}$$

In the second case, since $x - x_c < y - y_c$, by induction we have that $d_T(c,b) \leq (1+\sqrt{2})(y - y_c) + (x - x_c)$. Then, because $y < x$,

$$\begin{aligned} d_T(a,b) &\leq d_T(a,c) + d_T(c,b) \\ &\leq x_c + y_c + (1+\sqrt{2})(y-y_c) + (x-x_c) \leq (1+\sqrt{2})x + y \end{aligned}$$

Case 2: The interior of $R(a,b)$ is empty.

If no square $S_1, S_2, \ldots, S_k$ is inductive, $d_T(a,b) \leq (1+\sqrt{2})x + y$ by Lemma 6. Otherwise, let S_i be the first inductive square in the sequence and suppose that h_i is the inductive point of S_i. By Lemma 7, there is a j, $i \leq j \leq k$, such that $h_i, h_{i+1}, h_{i+2}, \ldots, h_j$ is a path in T of length at most $(x_{h_j} - x_{h_i}) + (y_{h_i} - y_{h_j})$ and such that $x - x_{h_j} \geq y_{h_j} - y \geq 0$. Since h_j is closer to b, using the L_∞-metric, than a is, we can apply induction to bound $d_T(h_j, b)$. Putting all this together with Lemma 6, we get:

$$\begin{aligned} d_T(a,b) &\leq d_T(a,h_i) + d_T(h_i,h_j) + d_T(h_j,b) \\ &\leq (1+\sqrt{2})x_{h_i} - (y_{h_i} - y) + (x_{h_j} - x_{h_i}) + (y_{h_i} - y_{h_j}) + (1+\sqrt{2})(x - x_{h_j}) + (y_{h_j} - y) \\ &\leq (1+\sqrt{2})x\ . \end{aligned}$$

If l_i is the inductive point of S_i, by Lemma 7 there is a j, $i \leq j \leq k$, such that $l_i, l_{i+1}, l_{i+2}, \ldots, l_j$ is a path in T of length at most $(x_{h_j} - x_{h_i}) + (y_{h_j} - y_{h_i})$ and such that $x - x_{h_j} \geq y - y_{h_j} \geq 0$. Because the position of j with respect to b is good and since l_j is closer to b, using the L_∞-metric, than a is, we can apply induction to bound $d_T(l_j, b)$. Putting all this together with Lemma 6, we get:

$$\begin{aligned} d_T(a,b) &\leq d_T(a,l_i) + d_T(l_i,l_j) + d_T(l_j,b) \\ &\leq (1+\sqrt{2})x_{l_i} + y_{l_i} + (x_{l_j} - x_{l_i}) + (y_{l_j} - y_{l_i}) + (1+\sqrt{2})(x - x_{l_j}) + (y - y_{l_j}) \\ &\leq (1+\sqrt{2})x + y\ . \end{aligned}$$

□

What remains to be done is to prove Lemma 6. To do this, we need to develop some further terminology and tools. Let x_i, for $1 \leq i \leq k$, be the horizontal distance between point a and the E side of S_i, respectively. We also set $x_0 = 0$.

Definition 8 *A square S_i has* potential *if*

$$d_T(a, h_i) + d_T(a, l_i) + d_{S_i}(h_i, l_i) \;\leq\; 4x_i$$

where $d_{S_i}(h_i, l_i)$ is the Euclidean distance when moving from h_i to l_i along the sides of S_i, clockwise.

Lemma 9 *If $R(a, b)$ is empty then S_1 has potential. Furthemore, for any $1 \leq i < k$, if S_i has potential but is not inductive then S_{i+1} has potential.*

Proof. If $R(a, b)$ is empty then, by Lemma 4, a lies on the W side of S_1 and x_1 is the side length of square S_1. Also, h_1 lies on the N or E side of S_1, and l_1 lies on the S or E side of S_1. Then $d_T(a, h_1) + d_T(a, l_1) + d_{S_1}(h_1, l_1)$ is bounded by the perimeter of square S_1 which is $4x_1$.

Now assume that S_i, for $1 \leq i < k$, has potential but is not inductive. Squares S_i and S_{i+1} both contain points l_i and h_i. Because S_i is not inductive, edge (l_i, h_i) must be steep and thus $d_x(l_i, h_i) < d_y(l_i, h_i)$. To simplify the arguments, we assume that l_i is to the W of h_i, i.e., $x_{l_i} < x_{h_i}$. The case $x_{l_i} > x_{h_i}$ can be shown using equivalent arguments.

Since $T_i = \triangle(h_{i-1}, h_i, l_{i-1} = l_i)$ or $T_i = \triangle(h_{i-1} = h_i, l_{i-1}, l_i)$, there has to be a side of S_i between the sides on which l_i and h_i lie, when moving clockwise from l_i to h_i. Using the constraints on the position of h_i and l_i within S_i from Lemma 4 and using assumptions that (l_i, h_i) is steep and that $x_{l_i} < x_{h_i}$, we deduce that l_i must be on the S side and h_i must be on the N or E side of S_i.

If h_i is on the N side of S_i then, because $x_{l_i} < x_{h_i}$, h_i must also be on the N side of S_{i+1} and either l_i is on the S side of S_{i+1} and

$$d_{S_{i+1}}(h_i, l_i) - d_{S_i}(h_i, l_i) \;=\; 2(x_{i+1} - x_i) \tag{1}$$

as shown in Figure 4a) or l_i is on the W side of S_{i+1}, in which case

$$d_{S_{i+1}}(h_i, l_i) - d_{S_i}(h_i, l_i) \;\leq\; 4(x_{i+1} - x_i) \tag{2}$$

as shown in Figure 4b).

If h_i is on the E side of S_i then, because $x_{i+1} > x_i$ (since (l_i, h_i) is steep), h_i must be on the N side of S_{i+1} and either l_i is on the S side of S_{i+1} and inequality (1) holds or l_i is on the W side of S_{i+1} and inequality (2) holds, as shown in Figure 4c).

Since S_i has potential, in all cases we obtain:

$$d_T(a, h_i) + d_T(a, l_i) + d_{S_{i+1}}(h_i, l_i) \;\leq\; 4x_{i+1}\ . \tag{3}$$

Assume $T_{i+1} = \triangle(h_i, h_{i+1}, l_i = l_{i+1})$; in other words, (h_i, h_{i+1}) is an edge of T with h_{i+1} lying somewhere on the boundary of S_{i+1} beween h_i and l_i, when moving clockwise from h_i to l_i. By the triangular inequality, $d_2(h_i, h_{i+1}) \leq d_{S_{i+1}}(h_i, h_{i+1})$ and we have that:

$$\begin{aligned}
d_T(a, h_{i+1}) + d_T(a, l_{i+1}) + d_{S_{i+1}}(h_{i+1}, l_{i+1}) \;&\leq\; d_T(a, h_i) + d_T(a, l_i) + d_2(h_i, h_{i+1}) + d_{S_{i+1}}(h_{i+1}, l_i) \\
&\leq\; d_T(a, h_i) + d_T(a, l_i) + d_{S_{i+1}}(h_i, l_i) \\
&\leq\; 4x_{i+1}\ .
\end{aligned}$$

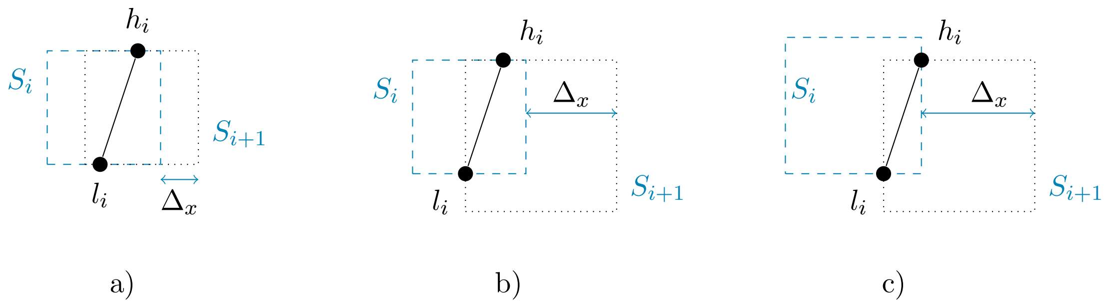

Figure 4: The first, second and fourth case in the proof of Lemma 9. In each case, the difference $d_{S_{i+1}}(h_i, l_i) - d_{S_i}(h_i, l_i)$ is shown to be at most $4\Delta_x$, where $\Delta_x = x_{i+1} - x_i$.

Thus S_{i+1} has potential. The argument for the case when $T_{i+1} = \triangle(h_i = h_{i+1}, l_i, l_{i+1})$ is symmetric. $\square$

Definition 10 *A vertex c (h_i or l_i) of T_i is* promising *in S_i if it lies on the E side of S_i.*

Lemma 11 *If square S_i has potential and $c = h_i$ or $c = l_i$ is a promising point in S_i then*

$$d_T(a, c) \leq 2x_c .$$

Proof. W.l.o.g., assume $c = h_i$. Since h_i is promising, $x_c = x_{h_i} = x_i$. Because S_i has potential, either $d_T(a, h_i) \leq 2x_{h_i}$ or $d_T(a, l_i) + d_{S_i}(l_i, h_i) \leq 2x_{h_i}$. In the second case, we can use edge (l_i, h_i) and the triangular inequality to obtain $d_T(a, h_i) \leq d_T(a, l_i) + |l_i h_i| \leq 2x_{h_i}$. $\square$

Here we define the maximal high and minimal low path.

Definition 12

- *If h_j is promising in S_j, the* maximal high path *ending at h_j is simply h_j; otherwise, it is the path $h_i, h_{i+1}, \ldots, h_j$ such that $h_{i+1}, \ldots, h_j$ are not promising and either $i = 0$ or h_i is promising in S_i.*
- *If l_j is promising in S_j, the* maximal low path *ending at l_j is simply l_j; otherwise, it is the path $l_i, l_{i+1}, \ldots, l_j$ such that $l_{i+1}, \ldots, l_j$ are not promising and either $i = 0$ or l_i is promising in S_i.*

Note that by Lemma 4, all edges on the path $h_i, h_{i+1}, \ldots, h_j$ are WN edges and thus the path length is bounded by $(x_{h_j} - x_{h_i}) + (y_{h_j} - y_{h_i})$. Similarly, all edges in path $l_i, l_{i+1}, \ldots, l_j$ are WS edges and the length of the path is at most $(x_{l_j} - x_{l_i}) + (y_{l_i} - y_{l_j})$.

We now have the tools to prove Lemma 6.

Proof of Lemma 6. If $R(a, b)$ is empty then, by Lemma 4, b is promising. Thus, by Lemma 9 and Lemma 11, if no square $S_1, \ldots, S_k$ is inductive then $d_T(a, b) \leq 2x < (1 + \sqrt{2})x + y$.

Assume now that there is at least one inductive square in the sequence of squares $S_1, \ldots, S_k$. Let S_j be the first inductive square and assume, for now, that h_j is the inductive point in S_j. By Lemma 9, every square S_i, for $i < j$, is a potential square.

Since (l_j, h_j) is gentle, it follows that $d_2(l_j, h_j) \leq \sqrt{2}(x_{h_j} - x_{l_j})$. Let $l_i, l_{i+1}, \ldots, l_{j-1} = l_j$ be the maximal low path ending at l_j. Note that $d_T(l_i, l_j) \leq (x_{l_j} - x_{l_i}) + (y_{l_i} - y_{l_j})$. Either $l_i = l_0 = a$ or l_i is a promising point in potential square S_i; either way, by Lemma 11, we have that $d_T(a, l_i) \leq 2x_{l_i}$. Putting all this together, we get

$$\begin{aligned} d_T(a, h_j) + (y_{h_j} - y) &\leq d_T(a, l_i) + d_T(l_i, l_j) + d_2(l_j, h_j) + y_{h_j} \\ &\leq 2x_{l_i} + (x_{l_j} - x_{l_i}) + (y_{l_i} - y_{l_j}) + \sqrt{2}(x_{h_j} - x_{l_j}) + y_{h_j} \\ &\leq \sqrt{2}x_{h_j} + x_{l_j} + y_{h_j} - y_{l_j} \\ &\leq (1 + \sqrt{2})x_{h_j} \end{aligned}$$

where the last inequality follows $x_{l_j} + y_{h_j} - y_{l_j} \leq x_{h_j}$, i.e., from the assumption that edge (l_j, h_j) is gentle.

If, instead, $c = l_j$ is the inductive point in inductive square S_j, let $h_i, h_{i+1}, \ldots, h_{j-1} = h_j$ be the maximal high path ending at h_j. Then $d_T(h_i, h_j) \leq (x_{h_j} - x_{h_i}) + (y_{h_j} - y_{h_i})$. Just as in the first case, we have that $d_T(a, h_i) \leq 2x_{h_i}$ and

$$\begin{aligned} d_T(a, l_j) - y_{l_j} &\leq d_T(a, h_i) + d_T(h_i, h_j) + d_2(h_j, l_j) - y_{l_j} \\ &\leq 2x_{h_i} + (x_{h_j} - x_{h_i}) + (y_{h_j} - y_{h_i}) + \sqrt{2}(x_{l_j} - x_{h_j}) - y_{l_j} \\ &\leq \sqrt{2}x_{l_j} + x_{h_j} + y_{h_j} - y_{l_j} \\ &\leq (1 + \sqrt{2})x_{l_j} \ . \end{aligned}$$

where the last inequality follows from $x_{h_j} + y_{h_j} - y_{l_j} \leq x_{l_j}$, i.e., from the assumption that (h_j, l_j) is gentle. □

4 Conclusion and perspectives

The L_1-Delaunay triangulation is the first type of Delaunay triangulation to be shown to be a spanner [Che86]. Progress on the spanning properties of the TD-Delaunay and the classical L_2-Delaunay triangulation soon followed. In this paper, we determine the precise stretch factor of an L_1- and L_∞-Delaunay triangulation and close the problem for good. The techniques we develop in this paper have potential to be successfully applied to Delaunay triangulations defined by other regular polygons, and possibly even to the classical Delaunay triangulation.

From a routing perspective, it is of interest to construct routes in geometric graphs that can be determined *locally* from a neighbor's coordinates only [BCD09]. Unfortunately, the route that is implicitely constructed in our proof is built using non-local decisions. It would be interesting to know whether in the L_1-Delaunay triangulation a route with stretch $\sqrt{4 + 2\sqrt{2}}$ can be constructed using a local routing algorithm. For TD-Delaunay triangulations, [BFvRV12] showed that there is no local routing algorithm that achieves a stretch that is less than $5/\sqrt{3} \approx 2.88$, whereas the stretch factor is actually 2. We leave open the questions regarding the gap between the stretch factor of L_1-Delaunay triangulations and the stretch that is possible using local routing.

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